Topology in physics 2019, exercises for lecture 8

- The hand-in exercises are exercises 1 and 2.
- Please hand it in electronically at `topologyinphysics2019@gmail.com` (1 pdf, readable!)
- Deadline is Wednesday April 10, 23.59.
- Please make sure your name and the week number are present in the file name.
- **ADDED REMARK:** Part of excercise 3 (no hand-in excercise) uses material that was not discussed in class yet. You can either try to see how far you get based on the lecture notes, or save this excercise for after lecture 10.
- **ADDED REMARK:** Utrecht students have midterms next week, so those students have an extra week to hand in this week's excercises, making the deadline April 17, 23.59 for them.

Exercises

⋆ Exercise 1: The holonomy group

Let $E \to M$ be a vector bundle with some fixed connection, and consider the set of all holonomies for loops $\gamma : [0,1] \to M$ that start and end at a given point $x \in M$:

$$\Gamma_x = \{G_\gamma | \gamma(0) = \gamma(1) = x\} \tag{1}$$

To be mathematically precise, assume that we are considering loops γ that are continuous and piecewise smooth (meaning that the loops may have occasional "corners" but are otherwise smooth).

a. Show that Γ_x is in fact a group.

b. Show that if M is connected, $\Gamma_x \cong \Gamma_y$ (as groups, not just sets) for any two $x, y \in M$.

The upshot is that for connected M, one can speak of *the* holonomy group of a bundle with connection; it turns out that this group often gives an interesting piece of data about the structure of the bundle.

⋆ Exercise 2: Chern-Simons theory

a. Show that the equation of motion for Chern-Simons theory is $F = 0$. That is, the "classical" solutions to Chern-Simons theory are the *flat* (zero curvature) gauge fields.

b. In Chern-Simons theory (on a trivial patch U), make a gauge transormation $A \to g^{-1}Ag + g^{-1}dg$. Show that the Chern-Simons Lagrangian changes as follows:

$$\delta L_{CS} = -d\mathrm{Tr}\ (dg \cdot g^{-1}A) - \frac{1}{3}\mathrm{Tr}\ \left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right) \qquad (2)$$

c. Use the above result to argue that if g is a map from a 3-manifold Σ without boundary into a Lie group, then

$$\frac{1}{24\pi^2}\int_\Sigma \mathrm{Tr}\ \left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right) \qquad (3)$$

is always an integer.

Exercise 3: Grassmann variables

If you are not familiar with Grassmann variables yet, this exercise contains some simple problem to get acquainted with those objects. In all that follows, θ_i $(1 \leq i \leq N)$ are Grassmann variables.

a. Show that the derivatives $\frac{d}{d\theta_i}$ and $\frac{d}{d\theta_j}$ anticommute.

b. Show that the anticommutator of $\frac{d}{d\theta_i}$ and θ_j equals δ_{ij} (the Kronecker delta).

c. If $f(\theta)$ and $g(\theta)$ each have given parity – that is, f contains only terms with either an even or an odd number of Grassmann variables, but not both, and similarly for g – give a Leibniz-rule-like expression for $\frac{d}{d\theta_i}(f(\theta)g(\theta))$.

d. As explained in the lecture, definite integration on a Grassmann algebra G should be implemented by a map $I : G \to G$ which is linear and satisfies the rules

1. $DI = 0$
2. $ID = 0$
3. $D(A) = 0 \implies I(BA) = I(B)A$

where D is the operator $\frac{d}{d\theta_i}$. Show that this means that $I = cD$ for some number c.